There are 7 questions with total 124 points in this exam.

1. Let $A \subset \mathbb{R}^n$.

   (a) (6 points) Define what it means to say that A has (n-dimensional) content $c(A)$ zero.

   Solution: A set $Z \subset \mathbb{R}^n$ has $n-$content zero if $\forall \epsilon > 0$, $\exists$ a finite set $\mathscr{C} = \{K_j\}_{j=1}^m$ of $n-$cells such that
   (a) $Z \subset \cup_{j=1}^m K_j$,
   (b) $\displaystyle\sum_{j=1}^m c(K_j) < \epsilon$.

   (b) (6 points) Define what it means to say that A has (n-dimensional) content.

   Solution: A set $A \subset \mathbb{R}^n$ is said to have content (or it is said to be (Jordan) measurable) if it is bounded and its boundary ∂A has content zero.

2. (a) (6 points) Let $A = (0,1) \times (0,1)$. Show that A has (2-dimensional) content and find its content.

   Solution: (1) For each $(x, y) \in A$, since $\|(x, y)\| \le \sqrt{2}$, A is bounded.
   (2) For any $4 > \epsilon > 0$, let $K_1 = [0, \frac{\epsilon}{4}] \times [0, 1]$, $K_2 = [1 - \frac{\epsilon}{4}, 1] \times [0, 1]$, $K_3 = [0, 1] \times [0, \frac{\epsilon}{4}]$, and $K_4 = [0, 1] \times [1 - \frac{\epsilon}{4}, 1]$.
   Since $\partial A = \big(\{0,1\} \times [0,1]\big) \cup \big([0,1] \times \{0,1\}\big) \subset \cup_{i=1}^4 K_i$, and $\displaystyle\sum_{i=1}^4 c(K_i) = \epsilon$, ∂A has content zero.
   By (1) and (2), we conclude that A has content and $c(A) = c(\bar{A}) = 1$.

   (b) (6 points) Let $Z = \{(x, y) \mid |x| + |y| = 1\}$. Show that Z has (2-dimensional) content zero.

   Solution: For each $\epsilon > 0$, choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\epsilon}{4}$. For each $i = 1, \ldots, n$, let $K_i = [\frac{i-1}{n}, \frac{i}{n}] \times [1 - \frac{i}{n}, 1 - \frac{i-1}{n}]$. Then $Z \cap \{(x, y) \mid x \ge 0, y \ge 0\} \subset \cup_{i=1}^n K_i$ and $\displaystyle\sum_{i=1}^n c(K_i) = \frac{1}{n} < \frac{\epsilon}{4}$. Similarly, we can cover the remaining part of Z by using cells of the same size. Hence, Z has content zero.

3. Let K be a closed cell in $\mathbb{R}^n$, f be a bounded function defined on K to $\mathbb{R}$, and $P = \{K_j\}_{j=1}^m$ be a partition of K.

   (a) (6 points) Define the upper sum $U_P(f, K)$, the lower sum $L_P(f, K)$, and a Riemann sum $S_P(f, K)$ of f corresponding to the partition P over K.

   Solution: Since f is bounded on K, $\sup_{K_j} f$ and $\inf_{K_j} f$ exist for each $j = 1, \ldots, m$.
   The upper sum of f corresponding to the partition P over K is defined by $U_P(f, K) = \displaystyle\sum_{j=1}^m \big(\sup_{K_j} f\big) c(K_j)$;
   the lower sum of f corresponding to the partition P over K is defined by $L_P(f, K) = \displaystyle\sum_{j=1}^m \big(\inf_{K_j} f\big) c(K_j)$; and

$S_P(f,K) = \sum_{j=1}^{m} f(x_j)c(K_j)$ for any $x_j \in K_j$ is called a Riemann sum of f corresponding to the partition P over K.

(b) (4 points) Let $Q = \{I_i\}_{i=1}^{l}$ be another partition of K. Define what it means to say that Q is a refinement of P.

Solution: We say that Q is a refinement of P if for each $I_i \in Q$, there exists a $K_j \in P$ such that $I_i \subseteq K_j$.

(c) (8 points) Let P, Q be partitions of K such that $P \subset Q$ i.e. Q is finer than P. Show that

$$L_P(f,K) \le L_Q(f,K) \le S_Q(f,K) \le U_Q(f,K) \le U_P(f,K).$$

Solution: For each $I_i \in Q$, since $P \subset Q$, there exists a $K_j \in P$ such that $I_i \subseteq K_j$.
Since $\inf_{K_j} f \le \inf_{I_i} f \le f(x_i) \le \sup_{I_i} f \le \sup_{K_j} f$,

and $c(K_j) = c\big(K_j \cap \bigcup_{i=1}^{l} I_i\big) = \sum_{I_i \in Q \text{ and } I_i \subseteq K_j} c(I_i)$,

we have $L_P(f,K) \le L_Q(f,K) \le S_Q(f,K) \le U_Q(f,K) \le U_P(f,K)$.

4. (a) (6 points) Let $f(x) = \begin{cases} 1 & \text{if } x \in [-1,0), \\ 2 & \text{if } x \in [0,1]. \end{cases}$

Show that f is integrable on $[-1,1]$ and find $\int_{-1}^{1} f(x)dx$.

Solution: For each $\epsilon > 0$, let $P = \{-1 = x_0 < x_1 < \cdots < x_n = 1\}$ be a partition of $[-1,1]$ such that $\|P\| = \max_{1\le i\le n} |x_i - x_{i-1}| < \epsilon$. Since $|U_P(f,[-1,1]) - L_P(f,[-1,1])| < 2\epsilon$, f is integrable on $[-1,1]$ by Riemann Criterion.

$\int_{-1}^{1} f(x)dx = \lim_{\|P\|\to 0} U_P(f,[-1,1]) = 1 + 2 = 3.$

(b) (6 points) For $a < b$, let $f(x) = \begin{cases} a & \text{if } x \in \mathbb{Q} \cap [0,1], \\ b & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$

Show that f is not integrable on $[0,1]$.

Solution: Let $\epsilon_0 = \dfrac{b-a}{2} > 0$ and P be any partition of $[0,1]$, since $|U_P(f,[0,1]) - L_P(f,[0,1])| = b - a > \epsilon_0$, f is not integrable on $[0,1]$.

(c) (6 points) Let A be a bounded subset of $\mathbb{R}^n$, and f be a bounded function defined on A to $\mathbb{R}$. Define what it means to say that f is integrable on A.

Solution: Let K be a closed cell in $\mathbb{R}^n$ containing A and f_K be an extension of f on K defined by $f_K(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \in K \setminus A. \end{cases}$

We say that f is integrable on A if f_K is integrable on K, and define $\int_A f = \int_K f_K$.

5. (a) (6 points) Let $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $(x,y) = \psi(r,\theta) = (r\cos\theta, r\sin\theta)$. Find $d\psi(r,\theta)$ and $\det d\psi(r,\theta)$.

Solution: $\det d\psi = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$

(b) (6 points) Let $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $(x, y, z) = \psi(\rho, \phi, \theta)$ $= (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi)$. Find $d\psi(\rho, \phi, \theta)$ and $\det d\psi(\rho, \phi, \theta)$.

Solution: $\det d\psi = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$
$= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$
$= \rho^2 \sin\phi\cos^2\phi(\cos^2\theta + \sin^2\theta) + \rho^2\sin^3\phi(\cos^2\theta + \sin^2\theta) = \rho^2\sin\phi.$

(c) (6 points) Show that $\int_0^\infty e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2}$.

Solution: $\left(\int_0^\infty e^{-x^2} dx\right)^2 = \int_0^\infty \int_0^\infty e^{-x^2-y^2} dxdy = \int_0^{\pi/2} \int_0^\infty re^{-r^2} drd\theta = \dfrac{\pi}{4}.$

(d) (6 points) Let $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = \rho^2\}$ be the ball of radius ρ with the center at origin.
Show that the volume of B is $\displaystyle\iiint_B dxdydz = \frac{4\pi\rho^3}{3}$.

Solution: $\displaystyle\iiint_B dxdydz = \int_0^{2\pi} \int_0^{\pi} \int_0^{\rho} r^2 \sin\phi \, drd\phi d\theta$
$= \left(\theta|_0^{2\pi}\right)\left(-\cos\phi|_0^{\pi}\right)\left(\dfrac{r^3}{3}|_0^{\rho}\right) = \dfrac{4\pi\rho^3}{3}.$

6. (a) (6 points) State the Bounded Convergence Theorem with which one can conclude that $\displaystyle\lim_{n\to\infty} \int_K f_n = \int_K \lim_{n\to\infty} f_n$.

Solution: Let $\{f_n\}$ be a sequence of integrable functions on a closed cell $K \subset \mathbb{R}^p$. Suppose that there exists $B > 0$ such that $\|f_n(x)\| \le B$ for each $n \in \mathbb{N}$ and for all $x \in K$. If the function $f(x) = \lim f_n(x)$, $x \in K$, exists and is integrable, then $\int_K f = \lim \int_K f_n$.

(b) (6 points) For $0 < a < 2$, show that $f_n(x) = e^{-nx^2}$ converges uniformly on $[a, 2]$, and show that $\displaystyle\lim_{n\to\infty} \int_a^2 f_n(x)\, dx = 0$.

Solution: For each $x \in [a, 2]$, since $\lim\limits_{n\to\infty} f_n(x) = 0$, and $\lim\limits_{n\to\infty} \sup\limits_{x\in[a,2]} |f_n(x)-0| \le \lim\limits_{n\to\infty} e^{-na^2} = 0$, f_n converges uniformly on $[a, 2]$. Since each f_n is continuous on $[a, 2]$, f_n is integrable there. Thus, $\{f_n\}$ is a sequence of integrable function that converges uniformly on $[a, 2]$ and we have $\displaystyle\lim_{n\to\infty} \int_a^2 f_n(x)\, dx = \int_a^2 \lim_{n\to\infty} f_n(x)\, dx = 0$.

(c) (6 points) Show that $f_n(x) = e^{-nx^2}$ does not converge uniformly on $[0, 2]$, and use a suitable convergence theorem to show that $\displaystyle\lim_{n\to\infty} \int_0^2 f_n(x)\, dx = 0$.

Solution: The limiting function f is given by $f(x) = \lim\limits_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \in (0, 2]. \end{cases}$
Since each f_n is continuous on $[0, 2]$ while f is not continuous at $x = 0$, the convergence of

f_n to f is not uniform on $[0, 2]$.
Note that $|f_n(x)| = e^{-nx^2} \le 1$ for each $n \in \mathbb{N}$ and for all $x \in [0, 2]$, and that f is integrable on $[0, 2]$, we can use the Bounded Convergence Theorem to conclude that $\lim_{n\to\infty} \int_a^2 f_n(x)\,dx = \int_a^2 \lim_{n\to\infty} f_n(x)\,dx = \int_0^2 f(x)\,dx = 0.$

7. (a) (8 points) Use the Dominated Convergence Theorem to show that the integral $F(t, u) = \int_0^\infty e^{-tx} \sin ux dx$ converges uniformly for all $t \ge \gamma > 0$ and for all $u \in \mathbb{R}$ to $\dfrac{u}{u^2 + t^2}$.

Solution: Since $|e^{-tx} \sin ux| \le e^{-tx}$ for all $x \ge 0$, $t \ge \gamma > 0$, $u \in \mathbb{R}$ and $\int_0^\infty e^{-tx} dx$ exists for all $t \ge \gamma > 0$, we can use the Dominated Convergence Theorem to conclude that the integral $F(t, u) = \int_0^\infty e^{-tx} \sin ux dx$ converges uniformly for all $t \ge \gamma > 0$ and for all $u \in \mathbb{R}$.
By using the integration by parts, we have $F(t, u) = \int_0^\infty e^{-tx} \sin ux dx = \dfrac{e^{-tx}}{-t} \sin ux|_0^\infty - \dfrac{u}{-t} \int_0^\infty e^{-tx} \cos ux dx = \dfrac{ue^{-tx}}{-t^2} \cos ux|_0^\infty - \dfrac{-u^2}{-t^2} \int_0^\infty e^{-tx} \sin ux dx = \dfrac{u}{t^2} - \dfrac{u^2}{t^2} F(t, u) = \dfrac{u}{u^2 + t^2}$.

(b) (6 points) Use the Dirichlet's test to show that the integral $G(t, u) = \int_0^\infty e^{-tx} \dfrac{\sin ux}{x} dx$ converges uniformly for all $t \ge \gamma > 0$ and for all $u \in \mathbb{R}$.

Solution: Since $|\int_0^c \sin ux\, dx| \le 2$ for all $c \ge 0$, $u \in \mathbb{R}$, and the function $\dfrac{e^{-tx}}{x}$ is monotone decreasing for $x > 0$, and $\lim_{x\to\infty} \dfrac{e^{-tx}}{x} = 0$, we can use the Dirichlet's test to conclude that $G(t, u) = \int_0^\infty e^{-tx} \dfrac{\sin ux}{x} dx$ converges uniformly for all $t \ge \gamma > 0$ and for all $u \in \mathbb{R}$.

(c) (8 points) Let $G(t) = \int_0^\infty e^{-x^2 - t^2/x^2} dx$ for $t > 0$. Show that $G'(t) = -2G(t)$ and $G(t) = \dfrac{\sqrt{\pi}}{2} e^{-2t}$.

Solution: $G'(t) = \int_0^\infty \dfrac{-2t}{x^2} e^{-x^2 - t^2/x^2} dx$
By setting $z = \dfrac{t}{x}$, we have $dz = -\dfrac{t}{x^2} dx$ and
$G'(t) = 2 \int_\infty^0 e^{-t^2/z^2 - z^2} dz = -2G(t)$
By separating the variables in the differential equation and integrating from 0 to t, we get $\log \dfrac{G(t)}{G(0)} = -2t \Rightarrow G(t) = G(0)\, e^{-2t} = \left(\int_0^\infty e^{-x^2} dx\right) e^{-2t} = \dfrac{\sqrt{\pi}}{2} e^{-2t}$.